conditions the antifoam promotes localized thinning and rupture of the bubble walls which cause coalescence to precede disengagement from the frit. Once formed, however, air bubbles rising through the antifoam-containing liquid show a high degree of resistance to agglomeration even after repeated collisions.

In general, antifoams are employed to break up stable surface foams and it is suggested that they function by rapidly spreading on the bubble surface, sweeping away surfactant, and thereby rupturing the bubble. They appear to be most effective against thin-walled, well-drained foams. Our experiments suggest that similar conditions exist on the frit surface where substantial drainage can occur between bubbles while they are still held in contact by attachment to the frit. In contrast, in the bulk liquid colliding bubbles can separate from each other faster than interfacial drainage can occur and the relatively thick liquid film maintains bubble integrity even when antifoam is present.

3. Hydrophobic frits modify the flow regime primarily by increasing the average bubble size prior to disengagement and not by promoting agglomeration during bubble formation. In this respect they behave much more like perforated plates than their hydrophilic counterparts and may be more appropriate to use in modeling studies.

ACKNOWLEDGMENT

The authors are much indebted to J. L. Henkes for his expertise in obtaining the photographs.

NOTATION

α = void fraction
γ = surface tension
ρ = liquid density
g = gravitational constant
J_g = drift flux

LITERATURE CITED

Ross, S., "Mechanisms of Foam Stabilization and Antifoaming Action," *Chem. Eng. Progr.*, 63, 9, 41 (1967).

Wallis, G. B., *One-Dimensional Two-Phase Flow*, Chap. 9, McGraw-Hill, New York (1969).

———., paper presented at Intern. Heat Transfer Conf., Boulder, Colo., ASME (1961).

Zuber, N., and J. Hench, "Steady State and Transient Void Fraction of Bubbling Systems and Their Operating Limits," Report #62GL100, General Electric Co., Schenectady, New York (1962).

Manuscript received September 28, 1973; revision received October 30 and accepted November 1, 1973.

Creeping Flow of a Power-Law Fluid over a Newtonian Fluid Sphere

V. MOHAN

Department of Chemical Engineering
Indian Institute of Technology, Madras-600036, India

In a paper in this journal, Nakano and Tien (1968) investigated the creeping flow of a power-law fluid over a Newtonian fluid sphere and presented an upper bound on the drag. A combination of Galerkin's method and variational principle due to Johnson (1960) was used in their analysis. However, the functional they minimized is proportional to the energy dissipation rate for the external fluid alone and does not draw any contribution from the inside fluid. It is to be expected that the minimizing of the functional obtained by viewing the system as a whole will lead to the true upper bound on the drag. Following this approach it is shown in this work that the upper bounds are, in fact, below those of Nakano and Tien for values of the relative viscosity factors around 1. Such flow situations arise in liquid-liquid extraction, phase separation, and allied processes (Treybal, 1963).

Adopting the notations of Slattery (1972), the equations of continuity and motion are

$$\frac{\partial \rho}{\partial t} = -(\rho\, v^j)_{,j} \tag{1}$$

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j\, v^i_{,j}\right) = -p^{,i} + \tau^{ij}_{,j} + \rho\, f^i \tag{2}$$

We make the same assumptions of flow as made by Nakano and Tien. The constitutive equations governing the flow are taken to be

$$\tau_l^k = 2\eta_i\, d_l^k \quad \text{(internal fluid)} \tag{3}$$

$$\tau_l^k = 2K\,(2\, d_j^i d_i^j)^{(n-1)/2}\, d_l^k \quad \text{(external fluid)} \tag{4}$$

The equation of motion for the inside fluid reduces to

$$D^4\psi_i = 0 \tag{5}$$

Let us define a function E as

$$E = \int_0^{\gamma^2} \eta(\gamma^2)\, d(\gamma^2) \tag{6}$$

where $\gamma^2 = d_i^j\, d_j^i$. It has been shown (Slattery, 1972) that for a homogenous function E,

$$tr\,(\tau \cdot D) = q\, E \tag{7}$$

where $q = n + 1$ for power-law fluids
$\quad\quad\;\, = 2$ for Newtonian fluids

We now seek to obtain the upper bound to the energy dissipation rate for the system comprising the internal and external fluid. Considering the function E for the actual velocity and a similar function E^* for a trial velocity profile, we may write as a consequence of the variational principles that (Slattery, 1972)

$$\int_{V_i} E_i \, dV \leq \int_{V_i} E_i^* \, dV + \int_{(S-S_v)_i} (v - v^*) \, [\tau - (p + \rho\phi) \, I] \cdot \underline{n} \, dS \quad (8)$$

$$\int_{V_o} E_o \, dV \leq \int_{V_o} E_o^* dV + \int_{(S-S_v)_o} (v - v^*) \, [\tau - (p + \rho\phi) \, I] \cdot \underline{n} \, dS \quad (9)$$

where $(S - S_v)$ is that part of the bounding surface where velocity is not explicitly specified. Combining Equations (7) to (9) and noting that $(S - S_v)$ is the fluid-fluid interface, we have

$$\frac{1}{2} \int_{V_i} tr \, (\tau \cdot D)_i \, dV + \frac{1}{n+1} \int_{V_o} tr \, (\tau \cdot D)_o \, dV = \int_{V_i} E_i \, dV + \int_{V_o} E_o \, dV \leq \int_{V_i} E_i^* \, dV + \int_{V_o} E_o^* \, dV \quad (10)$$

It can be seen that

$$\frac{n+1}{2} \int_{V_i} tr \, (\tau \cdot D)_i \, dV + \int_{V_o} tr \, (\tau \cdot D)_o \, dV \simeq \int_{V_i + V_o} tr \, (\tau \cdot D) \, dV \quad (11)$$

Since the contribution to the total drag from the internal fluid is often small, the approximation by Equation (11) is sufficient. From Equations (10) and (11),

$$\int_{V_i} tr \, (\tau \cdot D)_i \, dV + \int_{V_o} tr \, (\tau \cdot D)_o \, dV \leq (n+1) \left(\int_{V_i} E_i^* \, dV + \int_{V_o} E_o^* \, dV \right) \quad (12)$$

From an overall mechanical energy balance we can write

$$V_\infty F_d = \int_{V_i} tr \, (\tau \cdot D)_i \, dV + \int_{V_o} tr \, (\tau \cdot D)_o \, dV \quad (13)$$

where F_d is the drag force.

The problem of finding the upper bound to the drag force thus reduces to the minimization of the expression to the right of the inequality in Equation (12) subject to the differential Equation (5) and the set of boundary conditions

$$(v_r)_i = (v_r)_o = 0 \quad \text{at} \quad r = 1 \quad (14a)$$

$$(v_\theta)_i = (v_\theta)_o \quad \text{at} \quad r = 1 \quad (14b)$$

$$(\tau_{r\theta})_i = (\tau_{r\theta})_o \quad \text{at} \quad r = 1 \quad (14c)$$

$$\psi_o \longrightarrow -\tfrac{1}{2} \, r^2 \sin^2\theta \quad \text{as} \quad r \to \infty \quad (14d)$$

and $(v_r)_i$ and $(v_\theta)_i$ remain finite at $r = 0$ (14e)

Let us choose the trial stream functions to be

$$\psi_i = (C_1 \, r^2 + C_2 \, r^3 + C_3 \, r^4) \, (1 - z^2) \quad (15a)$$

$$\psi_o = (-\tfrac{1}{2} \, r^2 + A_1 \, r^\sigma + A_2/r) \, (1 - z^2) \quad (15b)$$

Applying the boundary conditions (14a) and (14b),

$$A_1 + A_2 = \tfrac{1}{2} \qquad C_1 + C_2 + C_3 = 0 \quad (16)$$

$$2C_1 + 3C_2 + 4C_3 = A_1\sigma - A_2 - 1 \quad (17)$$

The approximate solutions of Equations (5) and (14c) using Galerkin's method are obtained by setting

$$\int_0^1 \int_{-1}^1 r \, (1 - z^2) \, D^4\psi_i \, dz \, dr = 0 \quad (18)$$

$$\int_{-1}^1 [(\tau_{r\theta})_o - (\tau_{r\theta})_i] \Big|_{r=1} (1 - z^2) \, dz = 0 \quad (19)$$

Equations (18) and (19) reduce to

$$C_2 = 0 \quad (20)$$

and

$$\int_{-1}^1 (1 - z^2)^{3/2} \Big[6 \, C_3 \, X - (2 \, \overline{\gamma}_o^2)^{(n-1)/2}_{x=1} \, [A_1 \, (\sigma - 2) (\sigma - 1) + 6A_2] \Big] dz = 0 \quad (21)$$

where $\overline{\gamma}_o^2$ is the dimensionless second invariant of the rate-of-strain tensor given by

$$\overline{\gamma}_o^2 = x^4 \, [6z^2 \, [(2 - \sigma) \, A_1 \, x^{(1-\sigma)} + 6A_2 \, x^2]^2 + \frac{(1 - z^2)}{2} [(\sigma - 2)(\sigma - 1) \, A_1 \, x^{(1-\sigma)} + 6A_2 \, x^2]^2] \quad (22)$$

and

$$X = \eta_i / K (V_\infty/a)^{(n-1)} \quad (23)$$

Defining the drag coefficient C_d and Reynolds number Re as

$$\left. \begin{aligned} C_d &= \frac{2 \, F_d}{\pi \, a^2 \, \rho \, V_\infty^2} \\ Re &= \frac{(2a)^n \, V_\infty^{(2-n)} \, \rho}{K} \end{aligned} \right\} \quad (24)$$

and combining Equations (12) and (13) we have

$$Y = \frac{C_d \, Re}{24} \leq \frac{2^{(2n-1)/2}}{3} \int_{-1}^1 \int_0^1 (\overline{\gamma}_o^2)^{(n+1)/2} \, x^{-4} \, dx \, dz + \frac{2^{(n+2)}}{3} (n+1) \, C_1^2 \, X = F \quad (25)$$

NUMERICAL SOLUTION

Four of the six unknowns in Equations (15a) and (15b) may be eliminated using Equations (16), (17), and (20) to give

$$\left. \begin{aligned} A_2 &= \tfrac{1}{2} - A_1 \\ C_1 &= \tfrac{3}{4} - (\sigma + 1) \, A_1/2 \\ C_2 &= 0 \\ C_3 &= -C_1 \end{aligned} \right\} \quad (26)$$

A Fibonacci search (Mangasrian, 1972) on σ is performed to minimize F given by Equation (25). The value of A_1 for evaluating F is obtained by a numerical solution of the constraint Equation (21), for various values of σ used in the Fibonacci search. We now define G, the measure of the deviation from the boundary condition (14c), as

$$G = \int_{-1}^{1} [(\tau_{r\theta})_o - (\tau_{r\theta})_i]^2 \Big|_{r=1} dz/[K(V_x/a)^{n-1}]^2 \quad (27)$$

The values of G for various n for a typical value $X = 1$ are shown in Table 1. The consistently low values of G indicate that the boundary condition (14c) is very well satisfied. The minimum of F obtained by the Fibonacci search gives the upper bound to Y.

RESULTS

In Figure 1 are plotted the results of the present work and those of Nakano and Tien. The results of the present investigation are significantly below those of Nakano and Tien especially for X around 1. However, there is remarkable coincidence at extreme values of X. This is to be expected because the volume integral of E_i^* given by the second term on the right of the inequality 25 tends to zero as X tends to extreme values. It is observed that with increase in X from 0 to infinity, C_1 decreases rapidly to zero, and $C_1^2 X$ shows a maxima around $X = 1$.

Table 1. Measure of the Deviation $(\tau_{r\theta})_o - (\tau_{r\theta})_i$ at the Surface of the Sphere for $X = 1$

Flow behavior index, n	G defined by Equation (27)
0.9	1.20×10^{-5}
0.8	5.30×10^{-5}
0.7	5.80×10^{-5}
0.6	1.20×10^{-5}

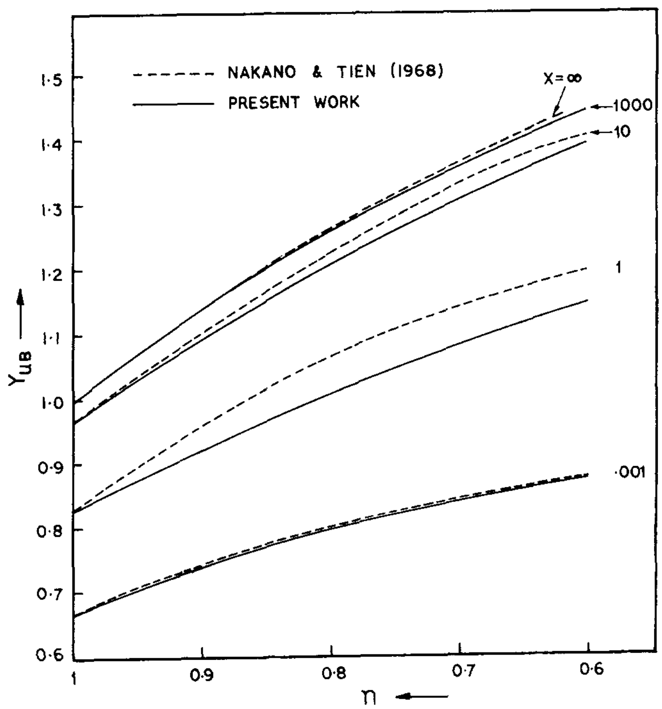

Fig. 1. Variation of the Upper Bound on Y with flow Behavior Index, n for various values of X: --------- Nakano and Tien (1968); ———————— present work.

CONCLUSION

The true upper bound to the drag on a fluid sphere placed in a flowing power-law fluid has been presented. There is considerable deviation from the results of Nakano and Tien for values of X around 1.

ACKNOWLEDGMENTS

The author wishes to acknowledge Dr. D. Venkateswarlu for his valuable comments, and the Indian Institute of Technology, Madras, and the Council of Scientific and Industrial Research, India, for the financial assistance.

NOTATION

a = radius of the fluid sphere
f^i = body force
n = flow behavior index
$\underline{n}$ = direction of the normal on the bounding surface
p = pressure
q = parameter in Equation (7)
r = dimensionless radius, R/a
t = time
v = velocity vector
x = $1/r$
z = $\cos \theta$
A_1, A_2, C_1, C_2, C_3 = constants in Equations (15a) and (15b)
C_d = drag coefficient
E = function defined by Equation (6)
F = defined in Equation (25)
F_d = drag force exerted on the sphere
G = defined by Equation (27)
K = consistency index
R = radial coordinate
V_∞ = terminal velocity
X = relative velocity factor defined by Equation (23)

Greek Symbols

γ^2 = second invariant of the rate-of-strain tensor
$\bar{\gamma}^2$ = dimensionless second invariant of the rate-of-strain tensor, $\gamma^2 (a/V_\infty)^2$
η_i = viscosity of the drop liquid
ρ = density of the drop liquid
σ = parameter in Equation (15b)
τ = extra stress tensor
θ = θ-coordinate
ψ = stream function

Subscripts

i = inside
o = outside

LITERATURE CITED

Johnson, M. W., Jr., "Some Variational Theorems for Non-Newtonian Flow," *Phys. Fluids,* **3**, 871 (1960).

Mangasrian, O. L., "Techniques of Optimisation," *Trans. Am. Soc. Mech. Engrs., J. Eng. Ind.,* **94**, Ser. B, 365 (1792).

Nakano, Y., and Tien Chi, "Creeping Flow of Power-Law Fluid over Newtonian Fluid Sphere," *AIChE J.,* **14**, 145 (1968).

Slattery, J. C., *Momentum, Energy and Mass Transfer in Continua,* pp. 259-68, McGraw Hill, N. Y. (1972).

Treybal, R. E., *Liquid Extraction,* 2nd edit., pp. 561-82, McGraw Hill, N. Y. (1963).

Manuscript received May 11, 1973; revision received July 19 and accepted July 20, 1973.